Simulation of Unsymmetrical 2-Phase Induction Machines

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Abstract—**The equations which describe the dynamic performance of unsymmetrical 2-phase induction machines are established, and an analog computer representation is developed from these equations. Computer representations of several single-phase machines including the split-phase, the capacitor-start, and the capacitor-start–capacitor-run types of induction motors are developed by extension and modification of the unsymmetrical 2-phase machine representation. In these simulations, provisions are made for stator switching during the starting period.**

The results of a computer study which demonstrates the free-acceleration characteristics of the capacitor-start, as well as those of the capacitor-start–capacitor-run single-phase induction motors, are given. Computer results which show the decay of the open-circuit terminal voltage and the instantaneous torque during an opening and reclosing of the main winding are also included.

Introduction

THE VERSATILITY of the analog computer in studying the dynamic behavior of symmetrical induction machines has been demonstrated in several publications [1]–[5]. Analog computer simulations of symmetrical induction machines which enable the engineer to investigate operating conditions which may be prohibitive to analyze or impractical to duplicate experimentally have been shown to be of practical importance [2], [3], [5]. Although different methods of simulating symmetrical induction machines are set forth in these references, analog computer simulations which may be used to study the transient performance of single-phase or unsymmetrical induction machines have not been given.

In order to apply the analog computer to advantage in this area, it is desirable to develop a computer representation which can easily simulate several types of single-phase induction machines. The theory of operation of the unsymmetrical 2-phase induction machine is applicable to a wide variety of single-phase induction machines. Therefore, the equivalent circuit and the computer representation of the unsymmetrical 2-phase induction machine will be developed and then modified and extended to describe the dynamic performance of various types of single-phase induction machines. In particular, equivalent circuits and computer representations are developed and computer results are given for the following types of motors:

1) Single-phase stator winding
2) Split-phase
3) Capacitor-start
4) Capacitor-start–capacitor-run.

In the simulation of the split-phase and capacitor-start types of motors, provisions are made for stator switching during the acceleration period. Also, a method of simulating an opening and reclosing of a stator phase is given and used to demonstrate the effects of opening and reclosing the main winding of a single-phase induction motor.

Basic Equations

A 2-phase machine with identical rotor windings and nonsymmetrical stator windings is commonly considered an unsymmetrical 2-phase induction machine. In the analysis of this type of machine, it is generally assumed that

1) Each stator winding is distributed to produce a sinusoidal mmf wave in space.
2) The rotor coils or bars are arranged so that, for any fixed time, the rotor mmf waves can be considered as space sinusoids having the same number of poles as the corresponding stator mmf wave.
3) The air gap is uniform.
4) The magnetic circuit is linear [6]–[10].

In some applications, all four assumptions may not be valid and it may be necessary to account for important features such as saturation or the harmonic content of the mmf waves. However, in many applications these idealizing assumptions, which are usually made for symmetrical and unsymmetrical induction machines, offer a convenient and sufficiently accurate means of predicting the transient and steady-state characteristics of induction machines [4], [9], [10]. The idealized machine is particularly useful in predicting the effects of the dynamic characteristics of an induction machine upon the overall response of a system in which it is incorporated.

The equations which describe the transient and steady-state performance of an idealized unsymmetrical 2-phase machine can be established by considering the elementary 2-pole machine shown in Fig. 1. Although it is unnecessary at this point in the development to consider specifically the single-phase application of this type of machine, the notation commonly used for single-phase induction motors will be employed where practicable.

Since it is assumed that each winding is distributed in such a way that it will produce a sinusoidal mmf wave, it is convenient to portray each winding as an equivalent single coil. The equivalent stator windings (Fig. 1) are in quadrature and are denoted as the m winding and the a winding. Although it may be convenient to consider these windings

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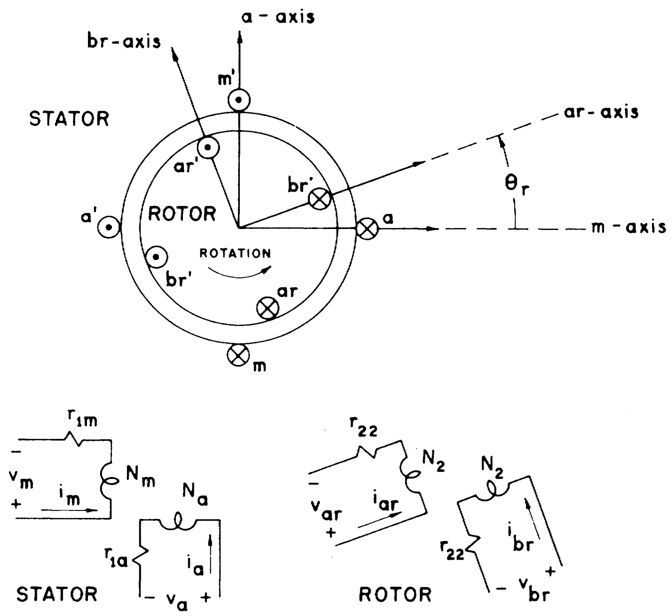

Fig. 1. Unsymmetrical 2-phase 2-pole induction machine.

as the main and auxiliary windings of a single-phase induction motor, it is unnecessary to affix such designations at this time. The stator windings are unsymmetrical; the windings have an unequal resistance and an unequal number of turns. The resistance and the effective number of turns of the m winding are denoted as r_{1m} and N_m, respectively. In the case of the a winding, r_{1a} and N_a denote the resistance and the effective number of turns. The rotor windings are in quadrature and are identical; that is, the windings have an identical number of effective turns N_2, and identical resistance r_{22}.

The stator voltage equations are written

$$v_m = p\lambda_m + i_m r_{1m} \tag{1}$$

$$v_a = p\lambda_a + i_a r_{1a}. \tag{2}$$

The rotor voltage equations are written

$$v_{ar} = p\lambda_{ar} + i_{ar} r_{22} \tag{3}$$

$$v_{br} = p\lambda_{br} + i_{br} r_{22}. \tag{4}$$

In the voltage equations just given, λ = the total flux linkages of a particular winding and p = the operator d/dt.

With the sinusoidally distributed windings portrayed as single equivalent coils, the mutual coupling between an equivalent stator coil and an equivalent rotor coil can be expressed as a sinusoidal function of the angular displacement between their magnetic axes. Therefore, the flux-linkage equations can be written as follows:

$$\lambda_m = L_m i_m + M_{m2} \cos \theta_r i_{ar} - M_{m2} \sin \theta_r i_{br} \tag{5}$$

$$\lambda_a = L_a i_a + M_{a2} \sin \theta_r i_{ar} + M_{a2} \cos \theta_r i_{br} \tag{6}$$

$$\lambda_{ar} = M_{m2} \cos \theta_r i_m + M_{a2} \sin \theta_r i_a + L_2 i_{ar} \tag{7}$$

$$\lambda_{br} = -M_{m2} \sin \theta_r i_m + M_{a2} \cos \theta_r i_a + L_2 i_{br} \tag{8}$$

where

θ_r = angular displacement between stator and rotor axes, expressed in electrical radians
L_m = self-inductance of m winding
L_a = self-inductance of a winding
L_2 = self-inductance of each rotor winding
M_{m2} = amplitude of the mutual inductance between the m winding and the identical rotor windings
M_{a2} = amplitude of the mutual inductance between the a winding and the identical rotor windings.

Transformation to an Axis of Reference Fixed in the Stator

In the case of a symmetrical machine, time-varying coefficients appear in the voltage equations as a result of the variation of the mutual inductances with respect to the displacement θ_r. These coefficients can be eliminated by transforming the voltages and currents of both the stator and the rotor to a common reference frame. When the machine is symmetrical, it is convenient to formulate a change of variables (transformation equations) which transforms the voltages and currents of both the stator and the rotor to an arbitrary reference frame [1], [6], [7]. The equations obtained by such a transformation can then be modified to give the equations which describe the behavior of the symmetrical induction machine in any specific reference frame.

If either the stator or the rotor of a machine is unsymmetrical, time-varying coefficients will appear in the voltage equations in all reference frames except the one fixed in the machine where the asymmetry exists. Therefore, in the case of an unsymmetrical 2-phase induction machine, it is convenient to select a reference frame fixed in the stator. A change of variables which will transform the stator and rotor voltages and currents to a reference frame fixed in the stator are expressed as follows:

Stator

$$f_{qs} = f_m \tag{9}$$

$$f_{ds} = -f_a. \tag{10}$$

Rotor

$$f_{qr} = f_{ar} \cos \theta_r - f_{br} \sin \theta_r \tag{11}$$

$$f_{dr} = -f_{ar} \sin \theta_r - f_{br} \cos \theta_r. \tag{12}$$

These equations of transformation can be correlated to the angular relation of the axes shown in Fig. 2. It is arbitrarily assumed that, at time 0, the m, ar, and q axes coincide. The variable f can represent either voltage, current, or flux linkage; for example,

$$v_{qs} = v_m \tag{13}$$

$$i_{qs} = i_m \tag{14}$$

$$\lambda_{qs} = \lambda_m. \tag{15}$$

It is clear that the transformation equations are valid, regardless of the form of the voltages and currents. With these equations of transformation there is, of course, a direct relationship between the actual stator variables and

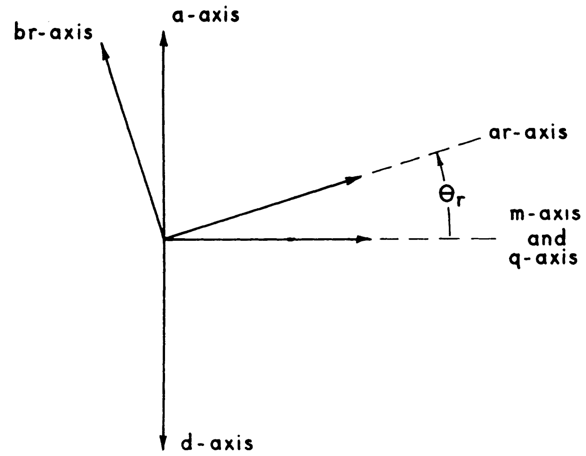

Fig. 2. Axes of unsymmetrical 2-phase 2-pole induction machine.

the ds and qs variables. Although this change in the notation of the stator quantities is not necessary, it is convenient in that it provides a d and q notation for both the stator and the rotor variables.

In the case of induction machines, a d–q axis does not imply an axis of reference fixed in the rotor as is the connotation with synchronous machines. However, when the frame of reference is fixed in the stator, the α and β subscripts may be preferable. If so, the d and q notation which is used in this development can readily be changed to an α and β notation. For example, subscript 1α might be used, instead of ds; 2α, instead of dr; 1β, instead of qs; and 2β, instead of qr.

If (9) through (12) are used to transform the voltages and currents of both the stator and the rotor to a reference frame fixed in the stator, the following equations are obtained:

$$v_{qs} = p\lambda_{qs} + r_{1m}i_{qs} \tag{16}$$

$$v_{ds} = p\lambda_{ds} + r_{1a}i_{ds} \tag{17}$$

$$v_{qr} = p\lambda_{qr} - \lambda_{dr}p\theta_r + r_{22}i_{qr} \tag{18}$$

$$v_{dr} = p\lambda_{dr} + \lambda_{qr}p\theta_r + r_{22}i_{dr} \tag{19}$$

where

$$\lambda_{qs} = L_m i_{qs} + M_{m2}i_{qr} \tag{20}$$

$$\lambda_{ds} = L_a i_{ds} + M_{a2}i_{dr} \tag{21}$$

$$\lambda_{qr} = L_2 i_{qr} + M_{m2}i_{qs} \tag{22}$$

$$\lambda_{dr} = L_2 i_{dr} + M_{a2}i_{ds}. \tag{23}$$

In the development of induction machine equivalent circuits, it is customary to refer all quantities to the stator windings. If the machine is symmetrical, the quantities can be referred to either stator winding by the same turns ratio. In the case of the unsymmetrical 2-phase machine, however, the stator windings do not have the same number of effective turns. Although in some instances it may be desirable to refer all quantities to one of the stator windings, in this development, the q quantities will be referred to the m winding and the d quantities will be referred to the a winding. This is a matter of preference and does not restrict the use of the resulting equivalent circuits. If all q quantities are referred to the m winding (N_m effective turns) and all d quantities are referred to the a winding (N_a effective turns), the voltage-equations can then be expressed as

$$v_{qs} = p\lambda_{qs} + r_{1m}i_{qs} \tag{24}$$

$$v_{ds} = p\lambda_{ds} + r_{1a}i_{ds} \tag{25}$$

$$v_{qr}' = p\lambda_{qr}' - \frac{N_m}{N_a}\lambda_{dr}'p\theta_r + r_{2m}i_{qr}' \tag{26}$$

$$v_{dr}' = p\lambda_{dr}' + \frac{N_a}{N_m}\lambda_{qr}'p\theta_r + r_{2a}i_{dr}' \tag{27}$$

where

$$\lambda_{qs} = L_{1m}i_{qs} + L_{Mm}(i_{qs} + i_{qr}') \tag{28}$$

$$\lambda_{ds} = L_{1a}i_{ds} + L_{Ma}(i_{ds} + i_{dr}') \tag{29}$$

$$\lambda_{qr}' = L_{2m}i_{qr}' + L_{Mm}(i_{qs} + i_{qr}') \tag{30}$$

$$\lambda_{dr}' = L_{2a}i_{dr}' + L_{Ma}(i_{ds} + i_{dr}') \tag{31}$$

in which

$$L_{Mm} = \frac{N_m}{N_2}M_{m2} \tag{32}$$

$$L_{Ma} = \frac{N_a}{N_2}M_{a2}. \tag{33}$$

where L_{1m} is the the leakage inductance of the m winding and L_{1a} is the leakage inductance of the a winding.

The following equations define the referred quantities in (24) through (31):

$$v_{qr}' = \frac{N_m}{N_2}v_{qr} \tag{34}$$

$$i_{qr}' = \frac{N_2}{N_m}i_{qr} \tag{35}$$

$$r_{2m} = \left(\frac{N_m}{N_2}\right)^2 r_{22} \tag{36}$$

$$L_{2m} = \left(\frac{N_m}{N_2}\right)^2 L_{22} \tag{37}$$

$$v_{dr}' = \frac{N_a}{N_2}v_{dr} \tag{38}$$

$$i_{dr}' = \frac{N_2}{N_a}i_{dr} \tag{39}$$

$$r_{2a} = \left(\frac{N_a}{N_2}\right)^2 r_{22} \tag{40}$$

$$L_{2a} = \left(\frac{N_a}{N_2}\right)^2 L_{22} \tag{41}$$

where L_{22} is the rotor leakage inductance. Also, it can be shown that

$$L_{Ma} = \left(\frac{N_a}{N_m}\right)^2 L_{Mm}. \tag{42}$$

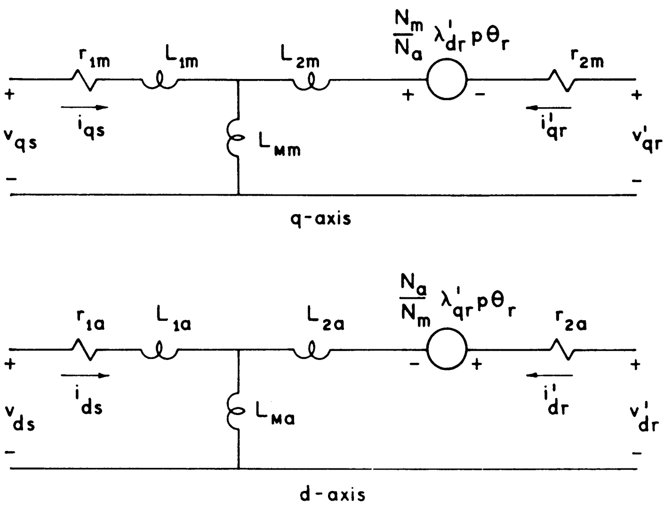

Fig. 3. Equivalent circuits (d and q) of unsymmetrical 2-phase induction machine, reference frame fixed in stator.

Ratio N_a/N_m appears in (27) and (42); its reciprocal appears in (26). When considering specifically a single-phase induction motor, it may be preferable to use the standard notation for this ratio; that is,

$$a = \frac{N_a}{N_m}. \tag{43}$$

Equations (24) through (31) suggest the equivalent circuit shown in Fig. 3. An expression for the instantaneous electromagnetic torque can be obtained by applying the principle of virtual displacement. This relation, which is positive for motor action, is expressed

$$T = \frac{P}{2}\left(\frac{N_a}{N_m}\lambda_{qr}'i_{dr}' - \frac{N_m}{N_a}\lambda_{dr}'i_{qr}'\right) \tag{44}$$

where P is the number of poles.

The transient and steady-state characteristics of an idealized unsymmetrical 2-phase induction machine are described by (24) through (31) and by (44). The complete dynamic characteristics are obtained by relating the electromagnetic torque, load torque, and speed ($p\theta_r$).

Computer Representation of an Unsymmetrical 2-Phase Induction Machine

It is convenient to develop the computer representation of the unsymmetrical 2-phase induction machine and then modify this representation to simulate various types of single-phase applications. The computer equations which can be used to simulate the unsymmetrical machine are obtained by first solving (28) through (31) for the currents; hence,

$$i_{qs} = \frac{1}{X_{1m}}(\psi_{qs} - \psi_{Mq}) \tag{45}$$

$$i_{ds} = \frac{1}{X_{1a}}(\psi_{ds} - \psi_{Md}) \tag{46}$$

$$i_{qr}' = \frac{1}{X_{2m}}(\psi_{qr}' - \psi_{Mq}) \tag{47}$$

$$i_{dr}' = \frac{1}{X_{2a}}(\psi_{dr}' - \psi_{Md}) \tag{48}$$

where

$$\psi_{Mq} = X_{Mm}(i_{qs} + i_{qr}') \tag{49}$$

$$\psi_{Md} = X_{Ma}(i_{ds} + i_{dr}'). \tag{50}$$

In these equations,

$$\psi_{qs} = \omega_e\lambda_{qs}\ldots \tag{51}$$

where ω_e is the base electrical angular velocity corresponding to rated frequency.

If (45) through (48) are substituted for the currents in (24) through (27) and in (49) and (50) and if (24) through (27) are then solved for ψ_{qs}, ψ_{ds}, ψ_{qr}', and ψ_{dr}', the following computer equations are obtained:

$$\psi_{qs} = \frac{\omega_e}{p}\left[v_{qs} + \frac{r_{1m}}{X_{1m}}(\psi_{Mq} - \psi_{qs})\right] \tag{52}$$

$$\psi_{ds} = \frac{\omega_e}{p}\left[v_{ds} + \frac{r_{1a}}{X_{1a}}(\psi_{Md} - \psi_{ds})\right] \tag{53}$$

$$\psi_{qr}' = \frac{\omega_e}{p}\left[v_{qr}' + \frac{N_m}{N_a}\frac{\omega_r}{\omega_e}\psi_{dr}' + \frac{r_{2m}}{X_{2m}}(\psi_{Mq} - \psi_{qr}')\right] \tag{54}$$

$$\psi_{dr}' = \frac{\omega_e}{p}\left[v_{dr}' - \frac{N_a}{N_m}\frac{\omega_r}{\omega_e}\psi_{qr}' + \frac{r_{2a}}{X_{2a}}(\psi_{Md} - \psi_{dr}')\right] \tag{55}$$

where

$$\psi_{Mq} = X_{qm}\left(\frac{\psi_{qs}}{X_{1m}} + \frac{\psi_{qr}'}{X_{2m}}\right) \tag{56}$$

$$\psi_{Md} = X_{da}\left(\frac{\psi_{ds}}{X_{1a}} + \frac{\psi_{dr}'}{X_{2a}}\right) \tag{57}$$

in which

$$X_{qm} = \frac{1}{(1/X_{Mm}) + (1/X_{1m}) + (1/X_{2m})} \tag{58}$$

$$X_{da} = \frac{1}{(1/X_{Ma}) + (1/X_{1a}) + (1/X_{2a})}. \tag{59}$$

In these equations, ω_r is the rotor speed in electrical radians per second.

Although the currents can also be eliminated from the torque expression, it is generally desirable to observe the four currents. Therefore, it is convenient to obtain the instantaneous torque by using

$$T = \frac{P}{2}\frac{1}{\omega_e}\left(\frac{N_a}{N_m}\psi_{qr}'i_{dr}' - \frac{N_m}{N_a}\psi_{dr}'i_{qr}'\right) \tag{60}$$

The computer representation is given in Fig. 4, where

$a = X_{qm}/X_{2m}$	$e = r_{2m}\omega_e/X_{2m}$
$b = X_{qm}/X_{1m}$	$f = N_m\omega_e/N_a$
$c = r_{1m}\omega_e/X_{1m}$	$g = 1/X_{1m}$
$d = \omega_e$	$h = 1/X_{2m}$

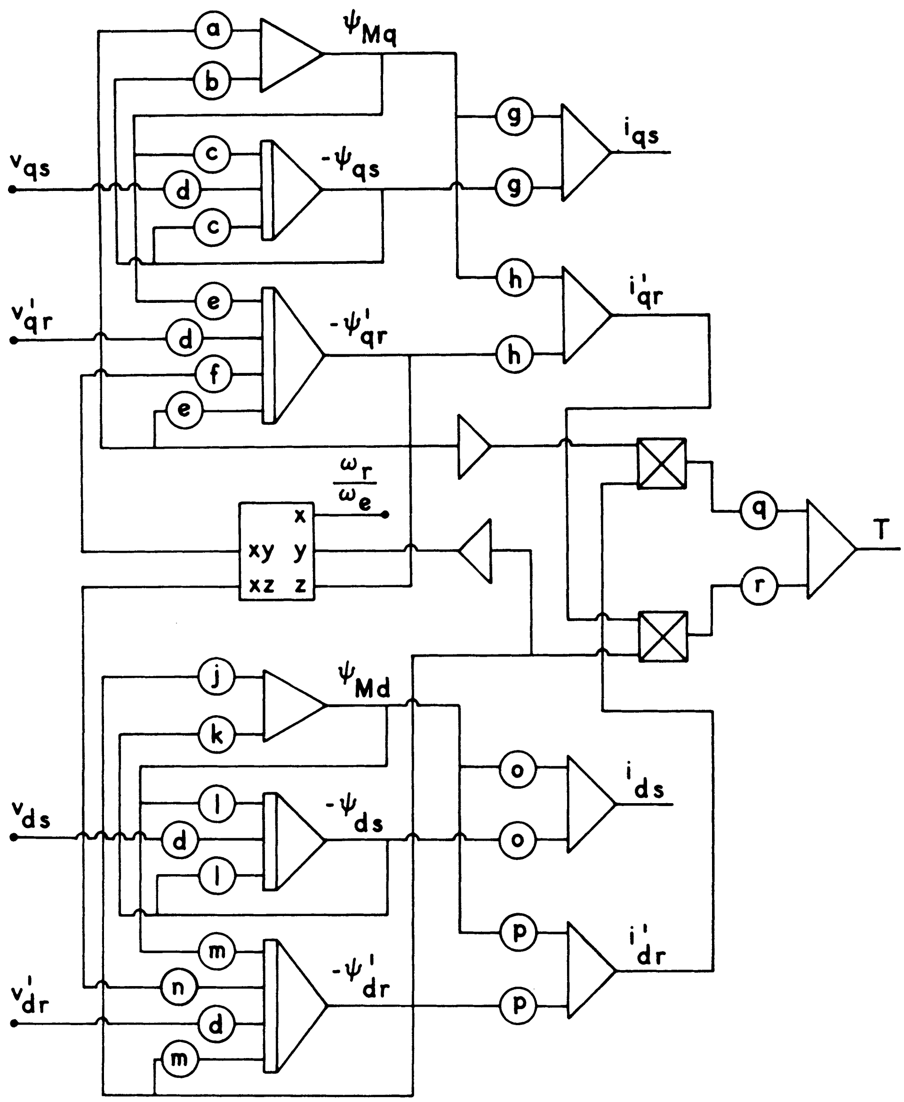

Fig. 4. Computer representation of unsymmetrical 2-phase induction machine.

$$j = X_{da}/X_{2a} \qquad n = N_a\omega_e/N_m$$
$$k = X_{da}/X_{1a} \qquad o = 1/X_{1a}$$
$$l = r_{1a}\omega_e/X_{1a} \qquad p = 1/X_{2a}$$
$$m = e = r_{2a}\omega_e/X_{2a} \qquad q = PN_a/2N_m\omega_e$$
$$r = PN_m/2N_a\omega_e.$$

No provision is made for possible sign inversion because of the multipliers. This representation can be used to study the transient and steady-state characteristics of the idealized unsymmetrical 2-phase induction machine. The simulation is applicable, regardless of the form or of the time relationships of the applied stator voltages. Moreover, if transformation equations (11) and (12) are implemented for the rotor-applied voltages, the simulation may be used to study the characteristics of a doubly fed unsymmetrical 2-phase induction machine. Saturation will not be considered in this development. However, the important effects of a nonlinear magnetic circuit may be incorporated into the computer simulation by direct extension and by modification of the methods which have been reported [4], [5], [11]–[13].

Although the computer representation is versatile, the simulation does not provide for the opening or reclosing of a stator phase. When this type of machine is used as a single-phase motor, one of the stator windings may be disconnected after the machine has reached 60 to 80 percent of synchronous speed. Therefore, in order to investigate the complete acceleration characteristics of this type of motor, it is necessary to simulate the opening of one of the stator windings.

If it is unnecessary to consider reclosing, the opening of a stator phase can be simulated by a simple modification of the computer representation given in Fig. 4. If, for example, the current i_{ds} $(-i_a)$ is zero, then (57) becomes

$$\psi_{Md} = \frac{X_{da}^*}{X_{2a}} \psi_{dr}' \tag{61}$$

where

$$X_{da}^* = \frac{1}{(1/X_{Ma}) + (1/X_{2a})}. \tag{62}$$

Therefore, an opening of the a winding at a normal current 0 (which may be interpreted as an opening of the auxiliary winding) can be represented by replacing, in the simulation, (57) with (61) at the instant i_{ds} becomes 0. The necessary switching can be implemented on the computer by incorporating either comparator or differential-type relays. With this method of simulation, a voltage will appear at the output of the amplifiers making up ψ_{ds} and i_{ds} after computer switching has occurred. These voltages are meaningless and can be disregarded, since neither is used in the simulation after switching. It is evident, therefore, that this particular method of simulation cannot be used to represent the reclosing of the stator phase.

A simulation which is applicable for both the opening and reclosing of a stator phase also can be obtained by modifying the basic machine setup. Assuming that a phase current ceases to flow after normal current 0, the modifications amount to maintaining the current at 0 by replacing the source voltage with the open-circuit voltage, in the computer simulation. If, for example, at the instant i_{ds} is 0, v_{ds} is replaced in the simulation by the voltage which appears across the mutual inductance, the current i_{ds} will be forced to remain at 0. The open-circuit voltage of the a winding is

$$v_{ds} = -v_a = \frac{p}{\omega_e} \psi_{ds}. \tag{63}$$

However, since i_{ds} is 0,

$$\psi_{ds} = \psi_{Md} = X_{Ma} i_{dr}'. \tag{64}$$

Therefore, the open-circuit voltage can be written

$$v_{ds} = \frac{X_{Ma}}{X_{Ma} + X_{2a}} \frac{p}{\omega_e} \psi_{dr}'. \tag{65}$$

Since $p\psi_{dr}'$ is available on the computer, (65) is obtained without differentiation. Therefore, an opening of the a winding can be simulated by incorporating a relay to switch v_{ds} from the source voltage to the open-circuit voltage, (65), at the instant i_{ds} becomes zero. With this method, the flux linkages and the current of the open-circuited winding are maintained correctly on the computer; the simulation is valid for reclosing.

The preceding development is given specifically for an opening and reclosing of the a winding of the stator; it is clear that an identical procedure can be used to simulate the opening and reclosing of the m winding. This method of representing the opening and reclosing of one or both of the stator phases can be used to investigate modes of operation other than stator switching during the starting period. For example, this simulation can be used to study the effects of reversing the polarity of a stator-applied voltage or to represent electronic switching devices which might be used in the stator phases.

Single-Phase Induction Machines

The basic equations and the computer representation for the unsymmetrical 2-phase induction machine have been established without regard to a specific application of this type of machine. Since the unsymmetrical 2-phase induction machine is designed primarily for single-phase applications, it is beneficial to stipulate any changes in the basic equations and in the computer representation which may be necessary to describe and investigate the behavior of several types of single-phase machines. In the following sections, the m winding and the a winding will be considered as the main winding and auxiliary winding, respectively.

Single-Phase Stator Winding

The equations which describe a single-phase induction machine with one stator winding can be derived by a procedure similar to that which was used for the unsymmetrical 2-phase machine or directly obtained from the equations developed for the unsymmetrical 2-phase machine. If the stator is equipped with only the main winding, the following voltage equations can be derived:

$$v_{qs} = p\lambda_{qs} + r_{1m} i_{qs} \tag{66}$$

$$v_{qr}' = p\lambda_{qr}' - \lambda_{dr}'' p\theta_r + r_{2m} i_{qr}' \tag{67}$$

$$v_{dr}'' = p\lambda_{dr}'' + \lambda_{qr}' p\theta_r + r_{2m} i_{dr}'' \tag{68}$$

where

$$\lambda_{qs} = L_{1m} i_{qs} + L_{Mm}(i_{qs} + i_{qr}') \tag{69}$$

$$\lambda_{qr}' = L_{2m} i_{qr}' + L_{Mm}(i_{qs} + i_{qr}') \tag{70}$$

$$\lambda_{dr}'' = L_{2m} i_{dr}'' + L_{Mm} i_{dr}''. \tag{71}$$

In this case, the torque can be expressed as

$$T = \frac{P}{2} (\lambda_{qr}' i_{dr}'' - \lambda_{dr}'' i_{qr}'). \tag{72}$$

In the preceding equations, the double prime is used to denote d quantities referred to the main winding. It is apparent, however, that since

$$v_{dr}' = \frac{N_a}{N_m} v_{dr}'' \tag{73}$$

$$i_{dr}' = \frac{N_m}{N_a} i_{dr}'' \tag{74}$$

$$\lambda_{dr}' = \frac{N_a}{N_m} \lambda_{dr}'' \tag{75}$$

therefore, (66) through (72) can also be obtained by deleting all ds variables in the equations for the unsymmetrical 2-phase induction machine and referring all quantities to the m winding.

This type of machine can be represented either by implementing (66) through (72) or by simulating an unsymmetrical 2-phase machine with provisions to hold the current in the auxiliary winding at zero ($i_{ds} = 0$). If motor operation is to be investigated, the rotor windings are short circuited. Thus,

$$v_{dr}' = v_{qr}' = 0 \tag{76}$$

$$v_{qs} = v_m \tag{77}$$

where v_m is the actual source voltage which is applied to the main winding of the machine.

Split-Phase Induction Motor

A split-phase induction motor can be simulated directly from the representation of the unsymmetrical 2-phase machine. The rotor windings are short circuited. Thus,

$$v_{dr}' = v_{qr}' = 0. \tag{78}$$

The stator windings are connected in parallel and, in order to obtain counterclockwise rotation of the rotor (see Fig. 1),

$$v_a = -v_m. \tag{79}$$

Thus,

$$v_{ds} = v_{qs} = v_m \tag{80}$$

where v_m is the actual source voltage which, in this case, is applied to both the main and auxiliary windings.

If the auxiliary winding is disconnected after the machine has reached 60 to 80 percent of synchronous speed, the switching can be simulated by one of the methods previously outlined.

Capacitor-Start and Capacitor-Start–Capacitor-Run Induction Motors

With a capacitor connected in series with the auxiliary winding, the following equations can be written:

$$v_a' = v_c + v_a \tag{81}$$

where

$$v_c = R_c i_a + \frac{\omega_e X_c}{p} i_a. \tag{82}$$

In these equations

v_c = the voltage across the capacitor,
v_a' = the voltage applied to the series combination of the capacitor and auxiliary winding, and
R_c = the resistance of the capacitor.

Since

$$v_{ds} = -v_a \tag{83}$$

$$i_{ds} = -i_a \tag{84}$$

and if

$$v_{ds}' = -v_a' \tag{85}$$

the voltage v_{ds} can be written

$$v_{ds} = v_{ds'} - R_c i_{ds} - \frac{\omega_e X_c}{p} i_{ds}. \tag{86}$$

The computer representation of an unsymmetrical 2-phase induction motor having a capacitor connected in series with the a winding is obtained by incorporating (86) with the equations for the unsymmetrical 2-phase induction machine. In the case of a capacitor-start induction motor, the rotor windings are short circuited; thus,

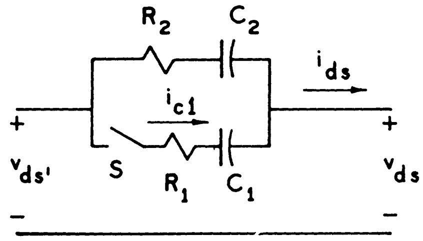

Fig. 5. Capacitor connections for capacitor-start–capacitor-run induction motor.

$$v_{dr}' = v_{qr}' = 0 \tag{87}$$

$$v_{ds}' = v_{qs} = v_m \tag{88}$$

where v_m is the actual source voltage which, in this case, is applied to the main winding and the series combination of the capacitor and the auxiliary winding.

It is clear that the capacitor is incorporated directly into the simulation [(86)] and, of course, can be varied without regard to transformation equations. The methods which have been outlined can be used to simulate the disconnecting of the auxiliary winding from the source.

The extensions necessary to include capacitor-start–capacitor-run induction motors are straight forward. In this type of machine, two capacitors are connected in parallel (Fig. 5). Resistances R_1 and R_2 are the resistances of capacitors C_1 and C_2, respectively. The switch S is provided to disconnect C_1 after the machine has accelerated to some predetermined speed. This switching can be simulated in detail by allowing the current i_{c_1} to control computer switching. However, in most cases it is sufficient to change the effective series resistance and capacitance at a normal phase current-zero (in this instance when $i_{ds} = 0$).

If the rotor windings of the circuits shown in Fig. 3 are short circuited and if the circuit shown in Fig. 5 is connected directly to the d-axis circuit of Fig. 3, the resulting circuits are the d- and q-axis equivalent circuits of a capacitor-start–capacitor-run induction motor with the frame of reference fixed in the stator; $v_{ds}' = v_{qs} = v_m$ is the actual applied voltage. The equivalent circuit can be used to study the transient and steady-state behavior of this type of machine and, since all parameters are incorporated directly, the equivalent circuit (or the computer representation of the equivalent circuit) can be used to show the effect of variation in any parameter.

Results of Computer Study

The results of an analog computer study are included to illustrate the facility of the computer representations which have been developed. Table I gives data about the machine which was simulated.

Table II gives the equivalent series resistance and reactance for a capacitor-start–capacitor-run operation. All resistances and reactances are expressed in ohms. The steady-state performance of this particular motor is considered in [9] and [10].

TABLE I
1/4-HP 110-VOLT 60-C/S 4-POLE MOTOR

Main Winding	$r_{1m} = 2.02$	$X_{Mm} = 66.8$	$r_{2m} = 4.12$
	$X_{1m} = 2.79$		$X_{2m} = 2.12$
Auxiliary Winding	$r_{1a} = 7.14$	$X_{Ma} = 92.9$	$r_{2a} = 5.74$
	$X_{1a} = 3.22$	$N_a/N_m = 1.18$	$X_{2a} = 2.95$

TABLE II

Capacitor-Start	Capacitor-Run
$R_c = 3.0$	$R_c = 9$
$X_c = 14.5$	$X_c = 172$

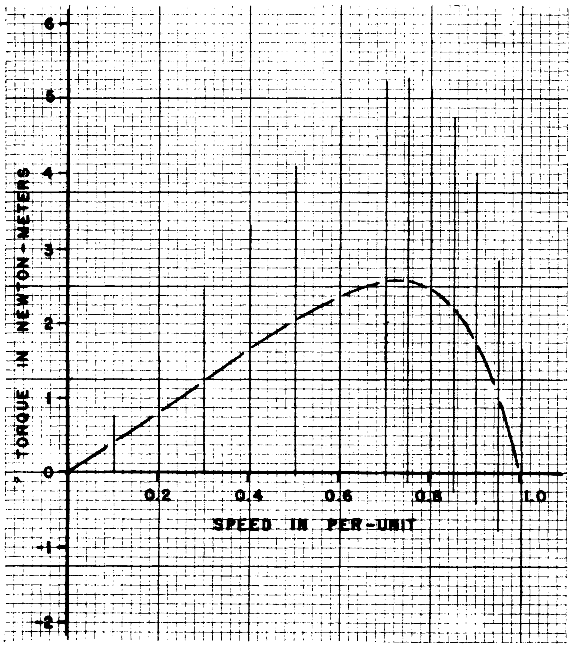

Fig. 6. Steady-state torque vs. speed characteristics for single-phase stator winding.

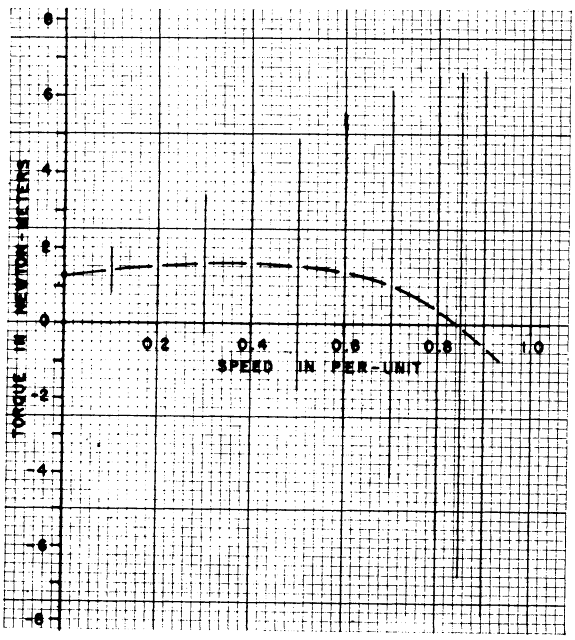

Fig. 7. Steady-state torque vs. speed characteristics for split-phase operation.

Steady-State Characteristics

The steady-state torque vs. speed characteristics for several types of stator connections are shown in Figs. 6 through 9. In particular, Fig. 6 shows the steady-state torque vs. speed characteristics with a single-phase stator winding (main winding); Fig. 7 shows the torque vs. speed characteristics for split-phase operation. Figures 8 and 9 show the steady-state torque vs. speed characteristics with fixed values of resistance and capacitance connected in series with the auxiliary winding. The characteristics in Fig. 8 are for the values of resistance and capacitance given for capacitor-start operation; the characteristics given in Fig. 9 are for the values given for capacitor-run operation.

At nonzero rotor speeds, the steady-state torque pulsates about an average value at twice the frequency of the applied voltages. The vertical lines in Figs. 6 through 9 are the computer traces of the pulsating torque. The average torque is given by the dashed lines which have been drawn on each computer recording.

Free-Acceleration Characteristics

In order to obtain the dynamic characteristics, it is necessary to relate torque and speed. Thus, since

$$T = \frac{2}{P} J p \omega_r + T_L \tag{89}$$

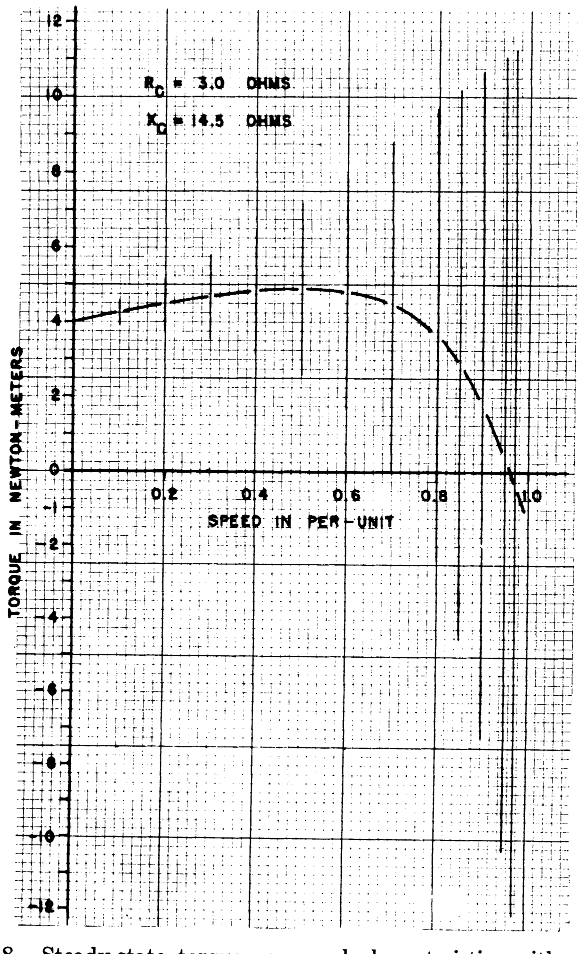

Fig. 8. Steady-state torque vs. speed characteristics with series capacitor equal to start capacitor.

where T_L is the load torque and J is the total inertia, the necessary computer equation is

$$\frac{\omega_r}{\omega_e} = \frac{1}{(2/P)\ J\omega_e p}(T - T_L). \tag{90}$$

In this study, the total inertia was selected as $J = 1.46 \times 10^{-2}$ kg·m².

Also, it was assumed that the applied voltage is expressed as

$$v_m = \sqrt{2}\ 110 \cos \omega_e t.$$

Figure 10 shows the torque vs. speed characteristics for free acceleration from the stall of the capacitor-start induction motor. In this case, the auxiliary winding was disconnected at the first current-zero after the machine had reached 75 percent of synchronous speed. Fig. 11 (a) shows the torque vs. time; Fig. 11 (b) shows the speed vs. time; and Fig. 11 (c) shows the voltage across the start-capacitor for this mode of operation. Similar characteristics are shown for the capacitor-start–capacitor-run induction motor in Figs. 12 and 13. In this study, the values of the equivalent series resistor and capacitor were changed from capacitor-start to capacitor-run at the first current 0 ($i_{ds} = 0$) after the machine had reached 75 percent of synchronous speed.

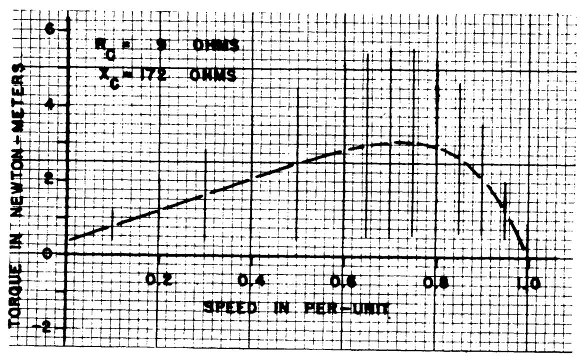

Fig. 9. Steady-state torque vs. speed characteristics with series capacitor equal to run capacitor.

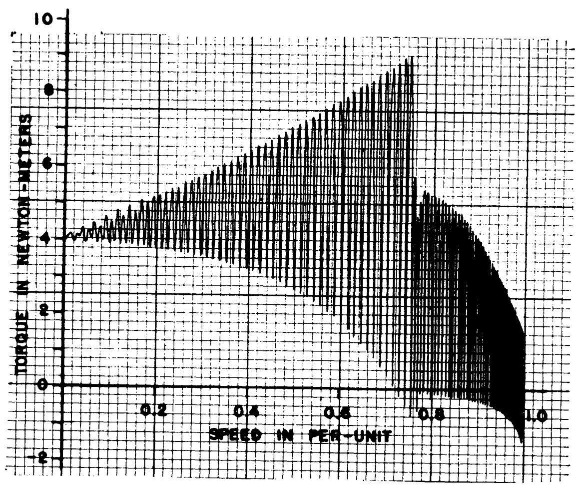

Fig. 10. Free-acceleration torque vs. speed characteristics for capacitor-start induction motor.

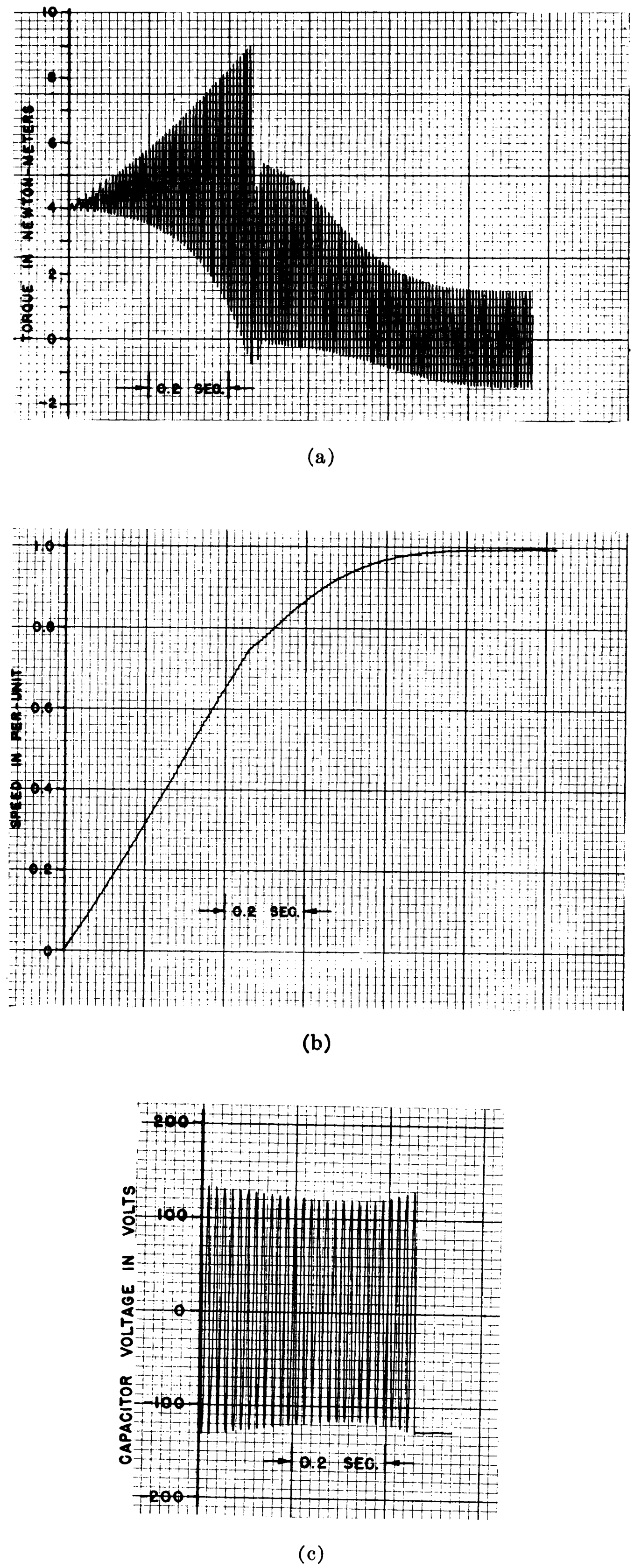

Fig. 11. Free-acceleration characteristics of capacitor-start induction motor. (a) Torque vs. time. (b) Speed vs. time. (c) Capacitor voltage vs. time.

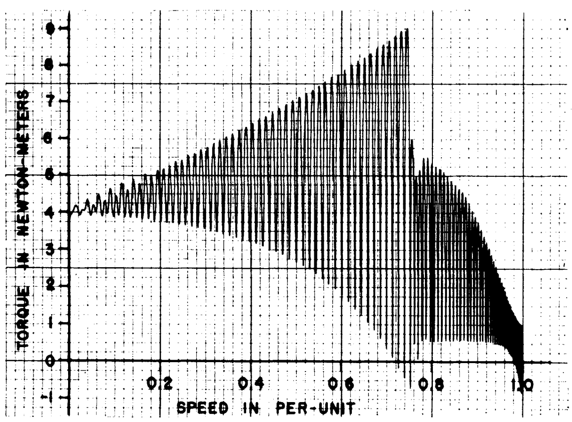

Fig. 12. Free-acceleration torque vs. speed characteristics for capacitor-start–capacitor-run induction motor.

Stator Phase Opened and Reclosed

In this study, the machine was allowed to establish steady-state operation at a fixed speed with only the main winding energized. With the speed held constant, the main winding was disconnected at a normal current 0 ($i_{qs} = 0$). Figures 14 and 15 show the decay of the open-circuit terminal voltage with the speed held constant at 0.4 pu (Fig. 14) and at 0.8 pu (Fig. 15).

Figures 16, 17, and 18 show the current and torque during an opening and subsequent reclosing of the main winding. For the results shown in Figs. 16 and 17, the speed was held fixed at 0.4 pu and the source voltage was reapplied in 10 ms (Fig. 16) and in 27 ms (Fig. 17) after the opening of the main winding. Similar results are shown in Fig. 18 for a constant speed of 0.8 pu.

Conclusion

The equations which describe the transient and steady-state behavior of unsymmetrical 2-phase induction machines have been set forth, and an analog computer representation has been developed from them. The equivalent circuits and the computer representation of the unsymmetrical 2-phase induction machine are quite general and can be readily modified to study a variety of applications of this type of machine. Although this facility has been demonstrated specifically for several single-phase induction motors, other perhaps less common modes of operation can also be investigated. For example, the equivalent circuits and the computer representation of the unsymmetrical 2-phase induction machine are valid, regardless of the form or the time relationships of the stator-applied voltages; generator action can be studied; and even the performance of a doubly fed unsymmetrical 2-phase induction machine with unbalanced rotor-applied voltages can be investigated.

The method of representing the opening and reclosing of one or both of the stator phases permits investigation of the performances of this type of machine when it is subjected to discontinuous applied-stator voltages. Such operating conditions may arise when the polarity of a stator-applied voltage is reversed, or when an electronic switching device is used in the stator phases.

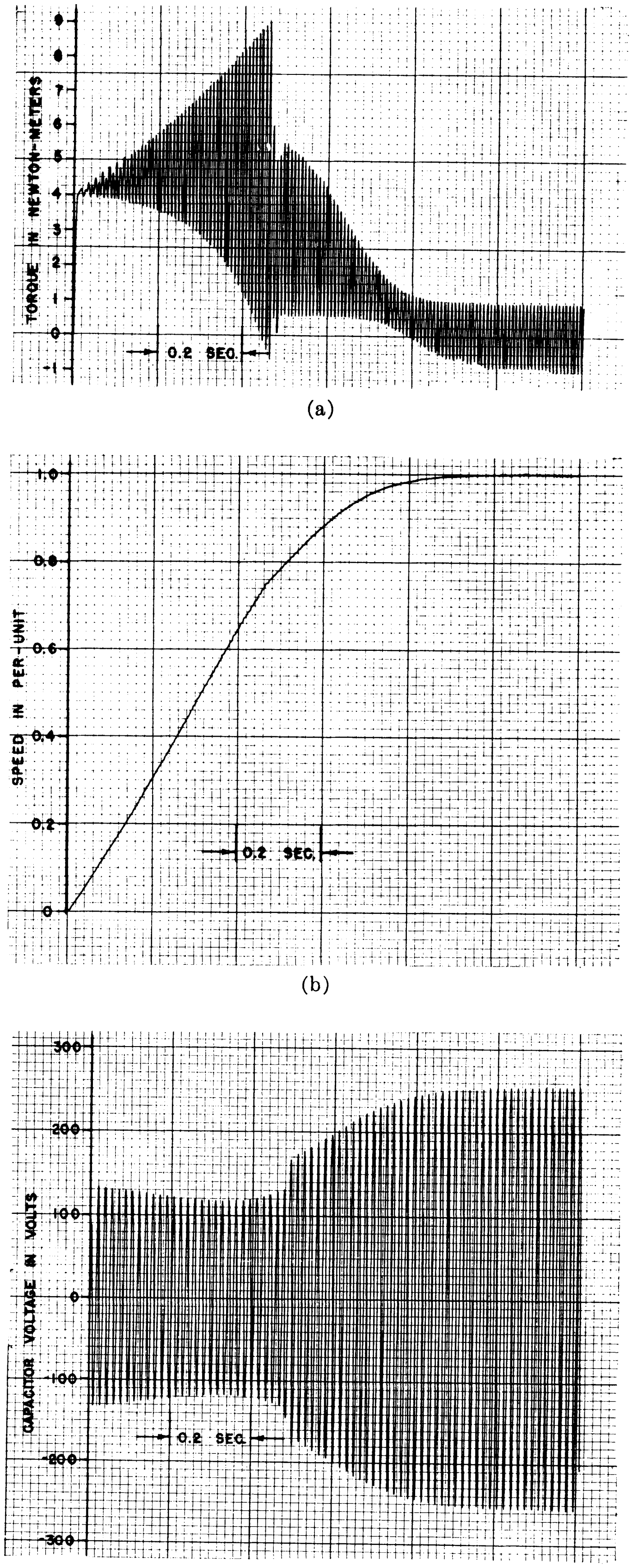

Fig. 13. Free-acceleration characteristics of capacitor-start–capacitor-run induction motor. (a) Torque vs. time. (b) Speed vs. time. (c) Capacitor voltage vs. time.

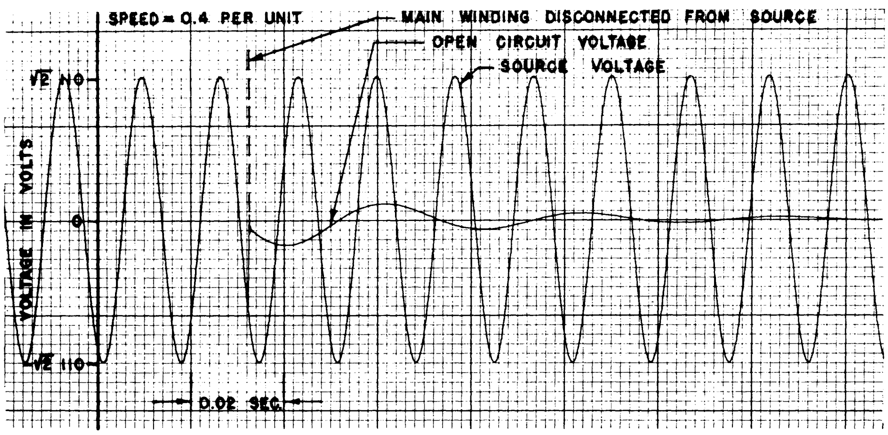

Fig. 14. Open-circuit voltage of main winding: speed = 0.4 pu.

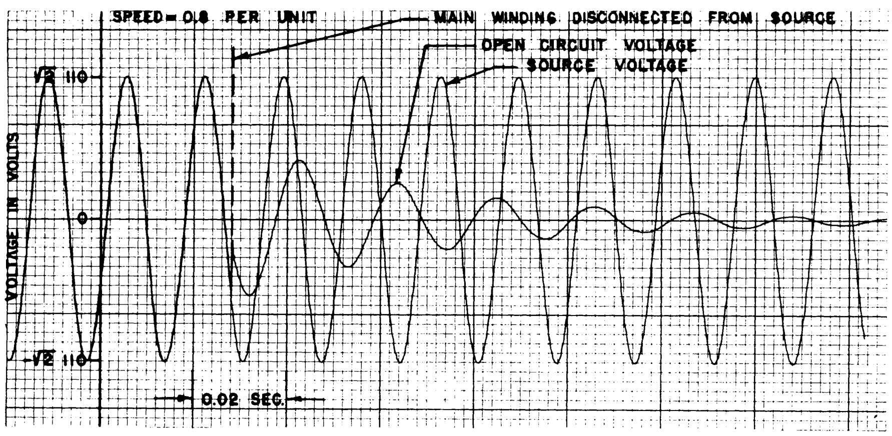

Fig. 15. Open-circuit voltage of main winding: speed = 0.8 pu.

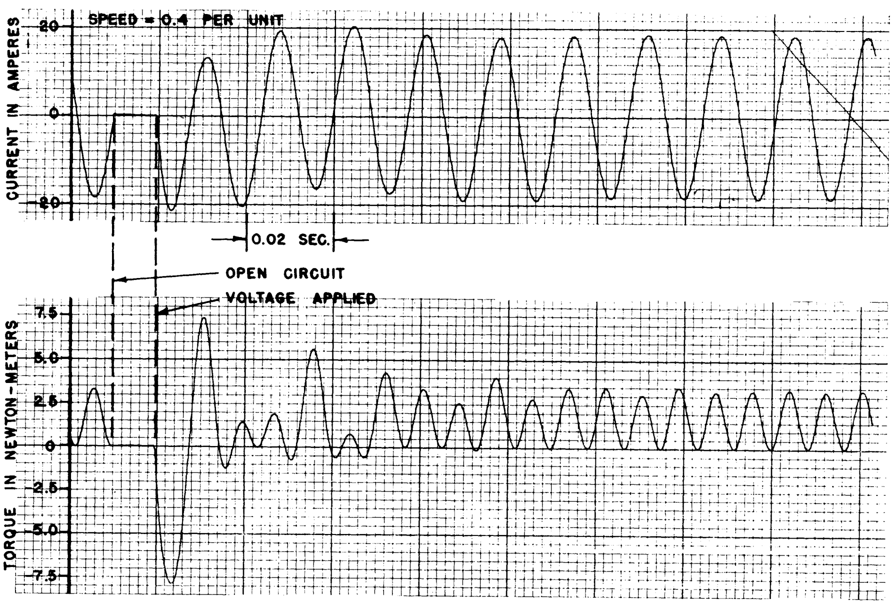

Fig. 16. Opening and reclosing of main winding: speed = 0.4 pu; reclosing in 10 ms.

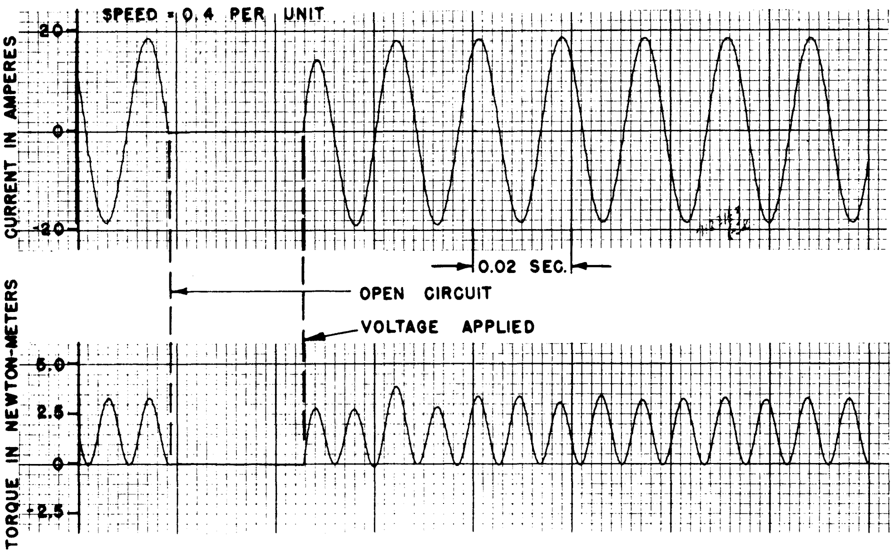

Fig. 17. Opening and reclosing of main winding: speed = 0.4 pu; reclosing in 27 ms.

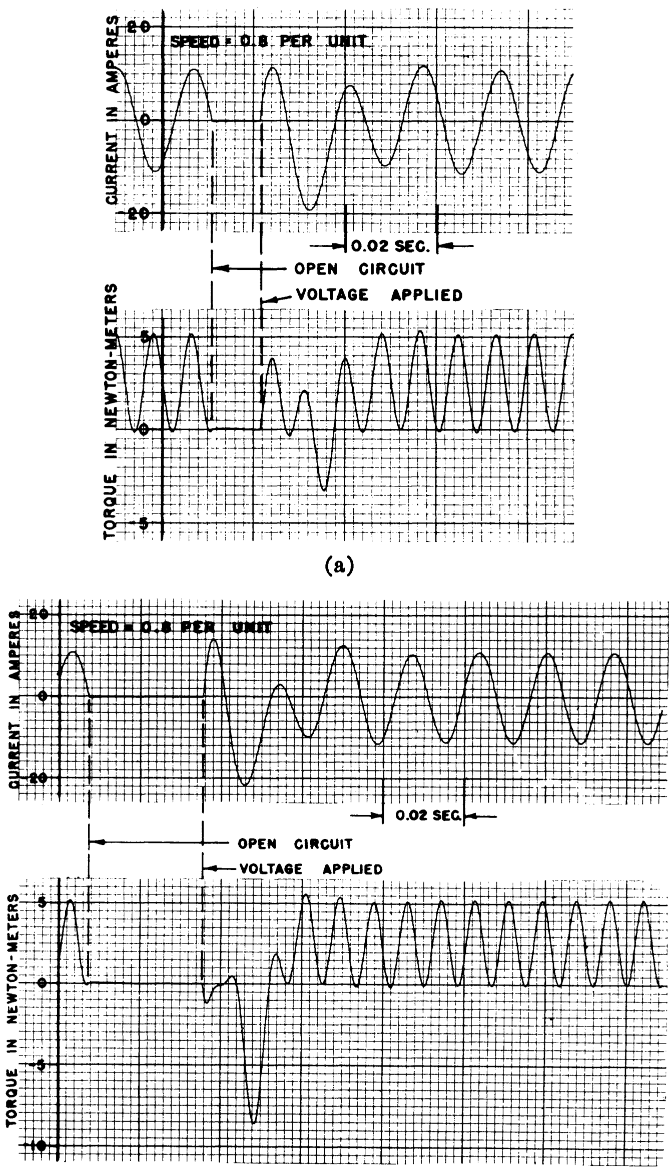

Fig. 18. Opening and reclosing of main winding: speed = 0.8 pu. (a) Reclosing in 10.5 ms. (b) Reclosing in 28 ms.

The use of the analog computer as an instructional aid to demonstrate the dynamic performance of various types of electric machines is of academic importance. The representations which have been developed for single-phase induction machine enable demonstration of the complete dynamic performance of an important and widely used type of electric machine.

ACKNOWLEDGMENT

The author wishes to thank Dr. C. H. Thomas and Dr. G. W. Staats of the Allis-Chalmers Manufacturing Company, Milwaukee, Wis., for their encouragement and assistance in revising the original manuscript. He is also grateful to the personnel of Allis-Chalmers who helped him prepare this manuscript for publication.

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Discussion

C. G. Veinott (Reliance Electric & Engineering Company, Cleveland, Ohio): This paper, together with the author's first reference, reminds us of 1) the growing importance of transient analyses of rotating machines, 2) the value of the analog computer in such a study, 3) the possibility of confirming the author's approach by correlating it with past steady-state analyses, and 4) the refreshing fact that the glamorous electronic developments of the last decade have not completely obliterated the interest of our universities in studying the problems of rotating machinery.

In Articles 10-7 and 12-8 of [9], the double-frequency pulsating torque is discussed and equations for calculating its magnitude, based on steady-state analysis are given. There, the pulsating torque is attributed to interactions between currents and fluxes rotating in opposite directions. Such analyses show that this pulsating torque can be substantially in excess of normal load torque, and Dr. Krause indicates this, too. These equations also show that the pulsating torque disappears when $Z_b = Z_f$ is at standstill. Dr. Krause's curves do show this zero pulsating torque at standstill.

I should like to ask the author if he attempted to compare the magnitudes of the pulsating torque he observed with values calculated by steady-state equations?

Manuscript received March 24, 1965.

P. C. Krause: I would like to thank Dr. Veinott for his comments. The equations he set forth in [9] (pp. 188–195) were used to calculate the magnitude of the pulsating torque of a single-phase induction motor at several rotor speeds. These calculated values of pulsating torque correspond to those shown in Fig. 6 of the paper.

Manuscript received March 25, 1965.